Discussing Malliavin Calculus and Monte Carlo Simulation to solve an Industry Question

(from a practitioner's perspective)

Dialid Santiago[1]

Industry Expert Lectures Series
Faculty of Natural and Mathematical Sciences
King's College London

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[1] This talk represents the views of the author alone, and not the views BofA Securities or any of her previous employers.

Agenda and Takeaways

- What is a Quant?
- What are price sensitivities (a.k.a. Greeks) and why do we care about them?
- What are the challenges in calculating sensitivities? Why numerical methods are needed?
- What is Malliavin Calculus? and why do we need it in Finance?

- Becoming a Quant is a cool career option that you may want to consider
- As a Quant you will have fun solving interesting problems by using the tools aquired during your studies. Particularly from Mathematics, Statistics, and Programming
- Today we will discuss and implement a method to estimate Greek sensitivities in an efficient manner

What is a Quant?

From Wikipedia

- Quantitative analysis is the use of mathematical and statistical methods[2] in finance and investment management. Those working in the field are quantitative analysts (quants).
- Although the original quantitative analysts were "sell side quants" from market maker firms, concerned with derivatives pricing and risk management, the meaning of the term has expanded over time to include those individuals involved in almost any application of mathematical finance, including the buy side.

[2] I would add computational/programming methods!

What is a Quant?

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- A quant is a colloquial term for a quantitative analyst, someone who applies mathematical and statistical methods[3] to financial and risk management problems. Quants work primarily in the finance industry, using their expertise to develop models and algorithms for trading, investment strategies, risk assessment, and other financial applications.
- They often have backgrounds in fields such as mathematics, statistics, physics, computer science, or engineering.

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- A quant (short for quantitative analyst) is a professional who uses mathematical, statistical, and computational techniques to analyze financial markets and develop trading strategies. Quants typically work in investment banks, hedge funds, asset management firms, and proprietary trading firms.

[3] again I would also stress the computational part

Quants in IB - How it Started

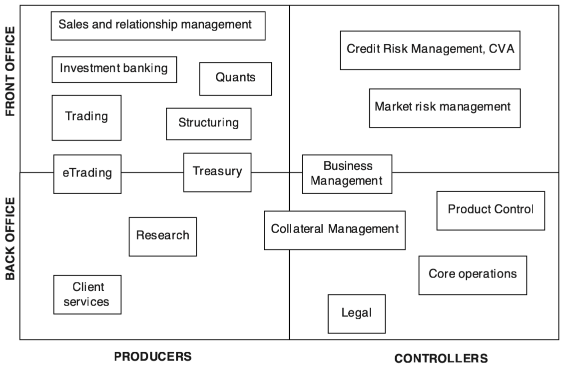


Figure 1.1 Divisions of an investment bank

Figure: Taken from "The Front Office Manual: The Definitive Guide to Trading, Structuring and Sales (Global Financial Markets) (2013)" by A. Sutherland, and J. Court.

Quant in IB - How it's Going

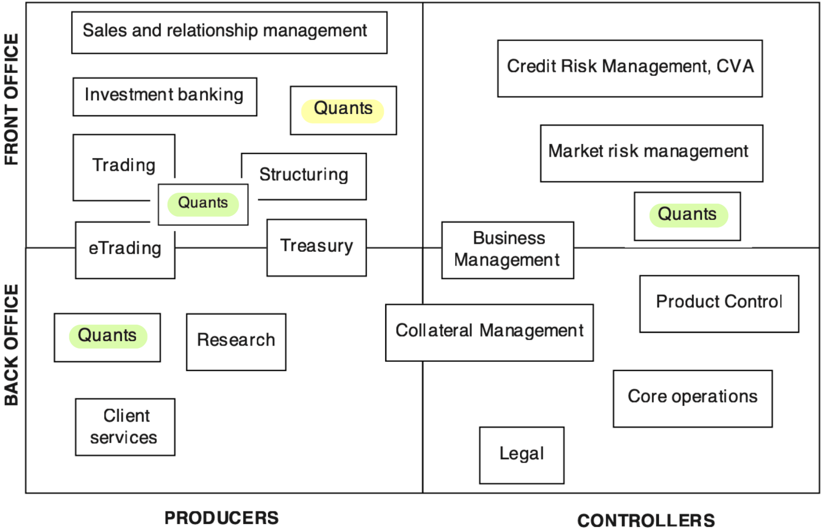


Figure 1.1 Divisions of an investment bank

*Image modified from original
to discuss the current Quant Teams presence in IB
@Quant_Girl

What are the Sensitivities/Greeks?[4]

- Price sensitivities, commonly referred to as "the Greeks", are a set of risk measures used in finance to quantify the sensitivity of the price of financial derivatives to changes in various underlying factors
- They tell us how the price of a derivative instrument, such as an option, changes in response to changes in factors such as the price of the underlying asset, time, volatility, and interest rates
- By understanding and managing the Greeks, traders and risk managers can develop effective strategies for pricing, hedging, and minimizing risk in complex financial markets

[4] and why do we care about them?

Mathematical Formulation

Under the risk neutral probability measure $\mathbb{Q}$, the present value of a derivative instrument with maturity T is computed as:

$$\text{Present Value}(t) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds}\Phi(\alpha, X_{t\leq T}(\beta)) \mid \mathcal{F}_t\right],$$

- Process Dynamics + Parameter Estimation (a.k.a. Calibration)
- Payoff Function[5]
- Discount Factor Term

Assuming a constant risk-free rate, and Markovianity of the process, we can work with the following simpler expression:

$$e^{-r(T-t)}\mathbb{E}\left[\Phi(\alpha, X_{t\leq T}(\beta)) \mid X_t = x\right] = u(T-t, r, \alpha, \beta, x).$$

Under this framework the idea of "sensitivity" can be expressed/captured by the mathematical concept of derivative.

[5] Note that it may depend on its own parameters represented by α.

Process Dynamics

- Let us consider a general Itô diffusion driven by the following SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

  with initial condition $X_0 = x$; where W denotes a standard Brownian motion in $\mathbb{R}^n$, and the coefficients b, σ are deterministic functions which satisfy enough regularity conditions to ensure the existence and uniqueness of the solution.
- We can also consider more complex dynamics, such as path-dependant, rough models, etc.

Note

- Choosing the model dynamics is a key step which cannot be separated from the Calibration process
- The calibration which consist of adjusting the parameters of a financial model so that its outputs closely match real-world market data. Essentially, it's like tuning a model so that it accurately reflects the current prices or behaviors of financial instruments, such as options, stocks, or bonds

Calibration Process

1. Choose a Model: You start with a mathematical model that describes how a financial asset behaves.
2. Get Market Data: Gather market data such as the current prices of options, stocks, bonds, or other derivatives.
3. Adjust Model Parameters: Adjust the parameters in your model so that the model's predicted prices are close to the observed market prices.
4. Test the Model: After calibration, you test the model in different aspects

Payoff Functions

- The payoff function of a financial instrument describes how much money you will gain or lose from owning that instrument, depending on certain conditions, like the price of an underlying asset at a specific time in the future
- It is essentially a formula or rule that tells you what you "get" at the end of a trade
- Call Option Payoff: A call option gives you the right (but not the obligation) to buy an asset (like a stock) at a specific price, called the strike price, on or before a certain date.
- Its payoff functions is given by

$$\Phi(X_T) = \max(X_T - K, 0),$$

  where X_T is the price of the asset at expiration (time T), and K is the Strike price (the price at which you can buy the asset).
- In general, we will consider a payoff function Φ for some financial instrument which depends on the price of the underlying X, at a finite number of points, i.e.;

$$\Phi(X) = \Phi(\alpha, X_{t_1}, \cdots, X_{t_n}),$$

  where Φ is inifinitely differentiable and all its partial derivatives have polynomial growth[6].

[6] We will see that these conditions can be relaxed

The Greeks

- Generally speaking, the Greeks are defined as

$$\text{Greek} = \frac{\partial}{\partial \bullet} u(T-t, r, \alpha, \beta, x) = \frac{\partial}{\partial \bullet} \mathbb{E}[e^{-r(T-t)} \Phi(\alpha, X_{t_1}^{\beta}, \cdots, X_{t_n}^{\beta}) | X_t = x],$$

  where the symbol $\bullet$, can be any variable for which we can take the derivative; e.g. the initial price x, the parameters α, β or the time t.

- Example: European call on the Black-Scholes dynamics (with volatility parameter σ) we get

$$\text{Greek} = \frac{\partial}{\partial \bullet} u(T-t, r, K, \sigma, x) = \frac{\partial}{\partial \bullet} \mathbb{E}[e^{-r(T-t)} (X_T(\sigma) - K)^+ | X_t = x],$$

Ok, now we know what the Greeks are... How do we calculate them?

- In particular, we are interested on:
  - $\Delta = \frac{\partial}{\partial x} u$
  - $\Gamma = \frac{\partial^2}{\partial x^2} u$
  - $\mathcal{V} = \frac{\partial}{\partial \sigma} u$
- Without loss of generality we are going to focus on Δ, i.e.:

$$\begin{aligned}\Delta &= \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\mathbb{E}[e^{-r(T-t)}(X_T(\sigma) - K)^+ | X_t = x + \epsilon] - \mathbb{E}[e^{-r(T-t)}(X_T(\sigma) - K)^+ | X_t = x]}{\epsilon}\end{aligned}$$

Traditional Methods I

- Finite Differences
  - Forward

$$\tilde{\Delta} \approx \frac{u(x+\epsilon) - u(x)}{\epsilon},$$

  - Backward

$$\tilde{\Delta} \approx \frac{u(x) - u(x-\epsilon)}{2\epsilon},$$

  - Central

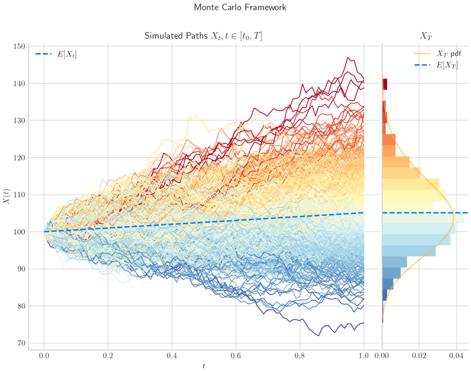
$$\tilde{\Delta} \approx \frac{u(x+\epsilon) - u(x-\epsilon)}{2\epsilon}.$$

Monte Carlo Fits Naturally here!

- Monte Carlo Simulation is used to estimate the value of financial instruments by simulating thousands (or even millions) of possible "scenarios"
- It relies on random sampling to model the uncertainty and randomness involved in financial markets, such as fluctuating stock prices, interest rates, or currency values
- In our case, Monte Carlo techniques fit perfectly because we want to estimate expectations of the form

$$\mathbb{E}[e^{-r(T-t)}\Phi(\alpha, X^{\beta}_{t_1}, \cdots, X^{\beta}_{t_n})|X_t = x]$$

So we just need to simulate and then take averages!



Traditional Methods II

- Pathwise Derivative

$$\Delta = \frac{\partial}{\partial x}\mathbb{E}[e^{-r(T-t)}\Phi(X_{t_1},\cdots,X_{t_n}) \mid X_t = x] = \mathbb{E}[e^{-r(T-t)}\frac{\partial}{\partial x}\Phi(X_{t_1},\cdots,X_{t_n}) \mid X_t = x]$$
$$= \mathbb{E}[e^{-r(T-t)}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Phi(X_{t_i})\frac{\partial X_{t_i}}{\partial x} \mid X_t = x].$$

- Likelihood Ratio

$$\Delta = \frac{\partial}{\partial x}\mathbb{E}[e^{-r(T-t)}\Phi(X_{t_1},\cdots,X_{t_n}) \mid X_t = x] = \mathbb{E}[e^{-r(T-t)}\Phi(X_{t_1},\cdots,X_{t_n})\pi \mid X_t = x],$$

with

$$\pi = \frac{\partial}{\partial x}\ln p(X_{t_1},\cdots,X_{t_n})$$

where $p(X_{t_1},\cdots,X_{t_n})$ is the density function of $(X_{t_1},\cdots,X_{t_n})$.

What are the Challenges?

- Numerical Stability: Estimating second-order Greeks like Gamma can be particularly challenging as it involves two levels of differentiation, amplifying numerical errors
- Computational Cost: Monte Carlo methods require a large number of simulations to reduce error, making them computationally expensive
- Time constrains: We have limited time to perform these calculations daily (more than once) on a massive scale
- Complexity of the Model: From a more theoretical point of view: What it we don't have enough smoothness?

QQ: Can Malliavin Calculus help us with some of these challenges? Yes!

What is Malliavin Calculus?

Figure: Paul Malliavin.

- It is known as the stochastic calculus of variations – extends the ideas of the calculus of variations from deterministic functions to stochastic processes
- It is named after French mathematician Paul Malliavin
- To tackle the problem, we need to familiarize ourselves with various concepts and classical results from Malliavin Calculus. In particular:
  - Malliavin Derivative
  - Chain Rule
  - Skorohod Integral
  - Integration by Parts

How to use Malliavin to Estimate Greeks?

Key Idea: Greek Calculation using Malliavin calculus

$$Malliavin\ Greek = \mathbb{E}\left[\underbrace{e^{-r(T-t)}}_{\text{Discount Factor}} \underbrace{\Phi(X_{t_1}, \cdots, X_{t_n})}_{\text{Payoff}} \underbrace{w}_{\text{Malliavin Weight}} \mid X_t = x\right],$$

where the Malliavin weight represents a random function to be determined which depends on the process and its first variation, i.e.:

$$\pi = H\left(X, \frac{\partial X}{\partial x}\right).$$

Bismut-Elworthy-Li (BEL) formula:

$$\Delta = \mathbb{E}\left[\underbrace{e^{-r(T-t)}}_{\text{Discount Factor}} \underbrace{\Phi(X_{t_1}, \cdots, X_{t_n})}_{\text{Payoff}} \underbrace{\delta(w^{\text{delta}})}_{\text{Malliavin Weight}} \mid X_t = x\right].$$

How to use Malliavin to Estimate Greeks?

Plan: Our plan consisted of 3 stages, namely

- Stage 1. Literature Review
- Stage 2. Implementation
- Stage 3. Benchmarking

Stage 1. Literature Review

- 20+ sources (papers, preprints, PhD theses, and books)
- The following ones deserve special mention

Foundation

- Fournié, E., Lasry, J. M., Lebuchoux, J., Lions, P. L., & Touzi, N. (1999). Applications of Malliavin calculus to Monte Carlo methods in finance. Finance and Stochastics, 3, 391-412.
- Fournié, E., Lasry, J. M., Lebuchoux, J., & Lions, P. L. (2001). Applications of Malliavin calculus to Monte-Carlo methods in finance. II. Finance and Stochastics, 5, 201-236.

Latest

- Alos, E., & Lorite, D. G. (2021). Malliavin calculus in finance: Theory and practice. Chapman and Hall/CRC.
- Giles, Michael B. "MLMC techniques for discontinuous functions." arXiv preprint arXiv:2301.02882 (2023).

Stage 2. Implementation

Case Studies → Monte Carlo Framework → Malliavin Weights Implementation

Dynamics	Payoff	Greeks
Black-Scholes	European	Delta
Heston	Asian	Gamma
SABR	Digital	Vega

- Note that the Malliavin weights depend on the model dynamics but not on the payoff
- I decided to use Python for quick prototyping and then C++ for production

Stage 3. Benchmarking

Accuracy and Variance → Computational Cost → Complexity

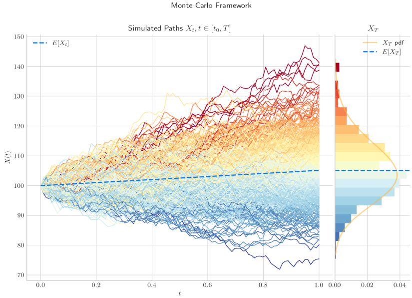
- Are there analytical or semi-analytical expression?

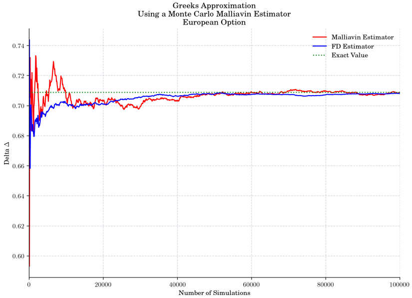
Dynamics / Payoff	European	Digital	Asian
Black-Scholes	Yes	Yes	No
Heston	Yes*	Yes*	No
SABR	No	No	No

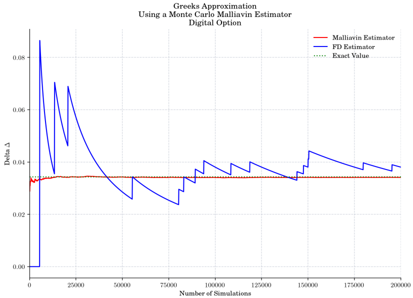
Table: Greeks closed analytical or semi-analytical (∗) form available.

- Finite Differences approach can be used as a benchmark in all cases

Key Results


Monte Carlo Framework
Simulated Paths $X_t, t \in [t_0, T]$
X_T
$E[X_t]$
X_T pdf
$E[X_T]$
$X(t)$
t


Greeks Approximation
Using a Monte Carlo Malliavin Estimator
European Option
Malliavin Estimator
FD Estimator
Exact Value
Delta Δ
Number of Simulations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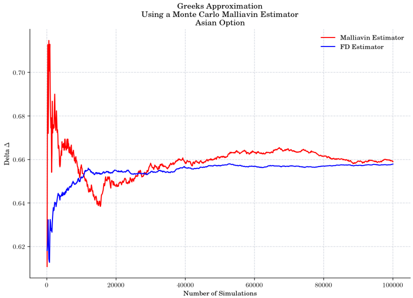
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Delta Δ
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Key Findings

- Accuracy: The Malliavin Weights method shows similar results, in terms of accuracy, to the Finite Differences approach – taking the closed-form as benchmark
- Variance: In the case of the European option, the Finite Differences method performs better than the Malliavin approach. This was a bit dissapointing!
  However, the Malliavin behaves better than Finite Differences in the case of Digital options.
- Computational Cost: the Malliavin Weights method is superior than the Finite Differences approach (cutting the time in half)
- Complexity: The Malliavin approach involves advanced mathematical concepts that may require additional training and expertise for implementation

Thank you for your attention!
Q&A